MAPS TO PROJECTIVE SPACE DEFINED BY GLOBAL SECTIONS OF INVERTIBLE SHEAVES

SARAH GRIFFITH

The following theorem is not proven in Hartshorne.

Theorem 0.1. *Let X be a scheme over a ring A.*

(1) *If $\phi : X \to \mathbb{P}^n_A$ is an A-morphism, then $\phi^*(\mathcal{O}(1))$ is an invertible sheaf on X, which is generated by the global sections $\phi^*(x_i), i = 0, \dots, n$.*
(2) *Let $\mathscr{L}$ be an invertible sheaf on X which is generated by nonzero global sections $s_0, \dots, s_n$. There exists a unique A-morphism $\phi : X \to \mathbb{P}^n_A$ for which there exists an isomorphism $\alpha : \mathscr{L} \to \phi^*(\mathcal{O}(1))$ with $\alpha(s_i) = \phi^*(x_i)$, and there is only one such α for this ϕ.*

It is convincingly handwaved, but the tools necessary to address it are not developed in the text (for example, what is $\phi^*(x)$? The "obvious" definition turns out not to be a definition at all, since pullback requires passing through two sheafifications). In this exposition, we present a complete proof.

The following results are, for the sake of brevity, largely stated in terms of global sections, but can be modified in obvious ways to deal with sections defined on open subsets.

1. Invertible ring elements and Nonvanishing Sections

Definition 1.1. Let X be a locally ringed space and $\mathcal{F}$ be an $\mathcal{O}_X$-module. For open $U \subseteq X$ and $x \in U$, we say that $s \in \Gamma(U, \mathcal{F})$ *vanishes at* x if $s_x \in \mathfrak{m}_x \mathcal{F}_x$. We say that s *vanishes* if there exists some $x \in U$ at which s vanishes, and that s is *nonvanishing* if there is no $x \in U$ at which s vanishes.

Lemma 1.2. *Let X be a ringed space and U be an open subset. Then $f \in \mathcal{O}_X(X)$ is invertible if and only if each f_x for $x \in X$ is invertible.*

Proof. The forward direction is obvious. Suppose f_x is invertible for each x. Let $\sigma : \mathcal{O}_X \to \mathcal{O}_X$ be the $\mathcal{O}_X$-module map given by multiplication by f. Then on each stalk, σ_x is an isomorphism, so σ is an isomorphism, and in particular some $\sigma_X(g) = 1$, i.e., f has an inverse. □

Corollary 1.3. *If X is locally ringed, then $f \in \mathcal{O}_X(X)$ is invertible if and only if f is nonvanishing.*

2. Pullbacks

Throughout this section, let $X \xrightarrow{\phi} Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_Y$-module and v be a global section of $\mathcal{F}$.

Remark 2.1. Passing to a stalk is functorial on the category of $\mathcal{O}_X$-modules. By virtue of being a colimit, it commutes with tensor products, commutes with sheafification, and commutes with other colimits.

Lemma 2.2. *We have $(\phi^*\mathcal{F})_x = \mathcal{F}_{\phi(x)}$ for each $x \in X$. Further, if $\zeta : \mathcal{F} \to \mathcal{G}$ is a morphism of $\mathcal{O}_Y$-modules, then $(\phi^*\zeta)_x = \zeta_{\phi(x)}$.*

Proof. The definition together with several applications of remark 2.1. □

The second statement in the next lemma is to be interpreted as saying that the constant function 1 pulls back to the constant function 1. This will allow us to manipulate pullbacks of sections of $\mathcal{O}_Y$-modules in terms of their relationship with the structure sheaf $\mathcal{O}_Y$.

Lemma 2.3. *There is a canonical isomorphism $\mathcal{O}_X \to \phi^*\mathcal{O}_Y$. Further, on a stalk at a point $x \in X$, this isomorphism takes $1 \mapsto 1 \otimes 1$.*

Proof. For any $\mathcal{O}_X$-module Q, we have natural isomorphisms

$$\mathrm{Hom}_{\mathcal{O}_X\text{-mod}}(\mathcal{O}_X, Q) \cong \Gamma(X, Q) = \Gamma(Y, \phi_*Q) \cong \mathrm{Hom}_{\mathcal{O}_Y\text{-mod}}(\mathcal{O}_Y, \phi_*Q) \cong \mathrm{Hom}_{\mathcal{O}_X\text{-mod}}(\phi^*\mathcal{O}_Y, Q)$$

The Yoneda lemma then tells us that the desired isomorphism exists and is found by setting $Q = \phi^*\mathcal{O}_Y$, taking $\mathrm{id}_{\phi^*\mathcal{O}_Y} \in \mathrm{Hom}_{\mathcal{O}_X\text{-mod}}(\phi^*\mathcal{O}_Y, \phi^*\mathcal{O}_Y)$, and following it back along the chain of isomorphisms. In $\mathrm{Hom}_{\mathcal{O}_Y\text{-mod}}(\mathcal{O}_Y, \phi_*\phi^*\mathcal{O}_Y)$, on an open set $U \subseteq Y$, we obtain a morphism induced by the following steps.

(1) Let $\tau_1 : \mathcal{O}_Y(U) \to \phi^{-1}\mathcal{O}_Y(\phi^{-1}(U))$ be the map of abelian groups induced by taking the relevant map into the direct limit (itself induced by restriction maps), then taking the map induced by sheafifying.
(2) Let $\tau_2 : \phi^{-1}\mathcal{O}_Y(\phi^{-1}(U)) \to \phi^{-1}\mathcal{O}_Y(\phi^{-1}(U)) \otimes_{\phi^{-1}\mathcal{O}_Y(\phi^{-1}(U))} \mathcal{O}_X(\phi^{-1}(U))$ be induced by taking maps of the form $x \mapsto x \otimes 1$, then sheafifying.

By lemma 2.2 for any $x \in U$ we have that $(\tau_2)_x$ sends $x \mapsto x \otimes 1$. Since taking colimits commutes, $(\tau_1)_x$ is the identity. Thus $(\tau_2\tau_1)_x$ is the map $x \mapsto x \otimes 1$. The image of $1 \in \Gamma(X, \mathcal{O}_X)$ under $\tau_2\tau_1$ therefore has germ $1 \otimes 1$ at each point of Y. The morphism $\mathcal{O}_X \to \phi^*\mathcal{O}_Y$ in the leftmost term of our sequence thus takes 1 to a section which (again by Lemma 2.2) has germ $1 \otimes 1$ at each point. □

Definition 2.4. We define the *pullback of v under ϕ*, denoted $\phi^*(v)$, as follows. There is a unique morphism $s : \mathcal{O}_Y \to \mathcal{F}$ taking $1 \mapsto v$, so since ϕ^* is functorial, we obtain a map $\phi^* s : \phi^*\mathcal{O}_Y \to \phi^*\mathcal{F}$. Since $\phi^*\mathcal{O}_Y$ is canonically isomorphic to $\mathcal{O}_X$, we obtain a canonical map $s' : \mathcal{O}_X \xrightarrow{\sim} \phi^*\mathcal{O}_Y \xrightarrow{\phi^* s} \phi^*\mathcal{F}$. Let $\phi^*(v) := s'(1)$.

Proposition 2.5. *For any $x \in X$, we have $\phi^*(v)_x = v_{\phi(x)} \otimes 1 \in \mathcal{F}_{\phi(x)} \otimes_{\mathcal{O}_{Y,\phi(x)}} \mathcal{O}_{X,x}$.*

Proof. We have that $\phi^*(v)_x$ is the evaluation at 1 of the map $\mathcal{O}_{X,x} \xrightarrow{\sim} (\phi^*\mathcal{O}_Y)_x \xrightarrow{\phi^*(s)_x} (\phi^*\mathcal{F})_x$. We know that the first map takes $1 \mapsto 1 \otimes 1$ and the second map takes $1 \otimes 1 \mapsto v_{\phi(x)} \otimes 1$, as desired. □

Corollary 2.6. *Suppose that additionally, X and Y are locally ringed spaces and that ϕ is a morphism of locally ringed spaces. Then we have that $\phi^*(v)$ vanishes at $x \in X$ if v vanishes at $\phi(x)$.*

Proof. Suppose that v vanishes at $\phi(x)$, say $v_x = mv'$ for some $m \in \mathfrak{m}_{\phi(x)}$ and $v' \in \mathcal{O}_{Y,\phi(x)}$. Then $\phi^*(v)_x = v_x \otimes 1 = \phi_x^\#(m)(v' \otimes 1)$. Since $\phi_x^\#$ is a local ring map, we have that $\phi^*(v)$ vanishes at x. □

The following result tells us that we can calculate a map of a section and then pull back, and this will be the same as (but presumably easier than) pulling back a section and then understanding the pulled back map.

Proposition 2.7. *Let $\zeta : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_Y$ modules. Then $\zeta^*(\phi^*(v)) = \phi^*(\zeta(v))$.*

Proof. These sections have the same stalks. □

The next result tells us that pulling back sections of the structure sheaf as a module is exactly the same as pulling them back as functions. We then proceed to turn this result into a definition.

Lemma 2.8. *Let $\tau : \mathcal{O}_X \to \phi^*\mathcal{O}_Y$ be the canonical isomorphism. Let $f \in \Gamma(Y, \mathcal{O}_Y)$. Then $\tau^{-1}(\phi^*(f)) = \phi^\#(f)$.*

Proof. For each $x \in X$, we have $\tau_x^{-1}(\phi^*(f)_x) = \tau_x^{-1}(f_{\phi(x)} \otimes 1) = \tau_x^{-1}(1 \otimes \phi^\#(f)_x) = \phi^\#(f)_x \tau^{-1}(1 \otimes 1) = \phi^\#(f)_x$. □

Definition 2.9. Henceforth we identify $\phi^*\mathcal{O}_Y$ with $\mathcal{O}_X$ along the canonical isomorphism. By Lemma 2.8, for each section $f \in \Gamma(Y, \mathcal{O}_X)$, we then have $\phi^*(f) = \phi^\#(f)$.

Proposition 2.10. *If $\mathcal{F}$ is generated by global sections $\{(x_i)_i\}$, then $\phi^*\mathcal{F}$ is generated by the global sections $\{(\phi^*(x_i))_i\}$.*

Proof. Let $\mathcal{O}_Y^{\oplus r_1} \to \mathcal{O}_Y^{\oplus r_1} \to \mathcal{F} \to 0$ be a presentation of $\mathcal{F}$. Since ϕ^* is a left adjoint, ϕ^* is right exact, so we obtain a presentation $\mathcal{O}_X^{\oplus r_1} \to \mathcal{O}_X^{\oplus r_1} \to \phi^*\mathcal{F} \to 0$, from which the result is clear. □

Corollary 2.11. *If $\mathcal{F}$ is quasicoherent, then $\phi^*\mathcal{F}$ has the same property.*

Proof. The proof is the same. □

It is not true that the pullback of a coherent sheaf must be coherent - this requires a locally Noetherian hypothesis. However, in the locally free case this does hold.

Proposition 2.12. *If $\mathcal{F}$ is locally free of constant rank, then $\phi^*\mathcal{F}$ is locally free of the same rank.*

Proof. Let $U \subseteq Y$ be an open subset on which there is an isomorphism $\psi : \mathcal{O}_Y|_U^{\oplus r} \to \mathcal{F}$. This pulls back to an isomorphism $\psi^* : \mathcal{O}_X|_{\phi^{-1}(U)}^{\oplus r} \to \phi^*\mathcal{F}|_{\phi^{-1}(U)}$. □

Proposition 2.13. *ϕ^* is an $\mathcal{O}_Y(Y)$-module homomorphism from $\Gamma(Y, \mathcal{F})$ to $\Gamma(X, \phi^*\mathcal{F})$.*

Proof. In other words, we must check that for each $v, w \in \Gamma(Y, \mathcal{F})$ and $f \in \Gamma(Y, \mathcal{O}_Y)$, we have $\phi^*(v + fw) = \phi^*(v) + \phi^*(f)\phi^*(w)$, and indeed they have the same stalks. □

In particular, for schemes over a field, the above result shows that pullback yields compatible vector space transformations for each open set of Y.

3. Trivializations

Throughout this section, let Y be a locally ringed space and $\mathscr{L}$ be an invertible sheaf on Y.

Definition 3.1. A *trivialization* of an $\mathcal{O}_Y$-module $\mathcal{F}$ is an isomorphism (in either direction) with $\mathcal{O}_Y^{\oplus r}$, for any r. A *local trivialization* is a trivialization of the restrictions of $\mathcal{F}$ and $\mathcal{O}_Y^{\oplus r}$ to some open subset.

A local trivialization of an invertible sheaf should be thought of as a local coordinate expression for the sections of that sheaf, which are thereby turned into functions (in fact, a local trivialization of any locally free sheaf can be thought of as turning sections into tuples of functions). Given a nonvanishing section s, we can use s to define local coordinate expressions for all the other sections by choosing to have s correspond to the constant function 1. We now make this notion rigorous.

Lemma 3.2. *Let $s \in \Gamma(U, \mathscr{L})$ be nonvanishing. Then taking the constant section 1 to s defines a local trivialization of $\mathscr{L}$ on U.*

Proof. We may as well assume $U = Y$. Let $\varphi : \mathcal{O}_Y \to \mathscr{L}$ be a morphism of $\mathcal{O}_Y$-modules defined by $1 \mapsto s$. For any $x \in Y$, let $\psi : \mathscr{L}|_U \xrightarrow{\sim} \mathcal{O}_Y|_U$ be a trivialization on some neighborhood U of

Y. Then $1 \mapsto \psi(s_x)$ defines the unique map τ making the following diagram commute

$$\begin{tikzcd} \mathcal{O}_{Y,x} \arrow[r, "\varphi_x"] \arrow[dr, "\tau"'] & \mathscr{L}_x \arrow[d, "\psi_x"] \\ & \mathcal{O}_{Y,x} \end{tikzcd}$$

If $\psi_x(s_x) \in \mathfrak{m}_x$, then $s_x = \psi^{-1}(\psi(s_x)) = \psi(s_x)\psi^{-1}(1) \in \mathfrak{m}_x\mathscr{L}_x$, a contradiction. Thus $\psi(s_x)$ is a unit, so τ is an isomorphism, whence $\psi_x^{-1}\tau = \varphi_x$ is an isomorphism. Since φ is an isomorphism on stalks, φ is an isomorphism. □

Lemma 3.3. *Let $\alpha : \mathscr{L} \to \mathcal{O}_Y$ be a trivialization. Let N be the set of nonvanishing global sections of $\mathscr{L}$ and K be the set of invertible global sections of $\mathcal{O}_Y$. Then $\alpha|_N$ yields a bijection between N and K.*

Proof. We will show $\alpha|_N : N \to K$ and $\alpha^{-1}|_K : K \to N$ are well defined. In this case we will already know they are inverses.

Suppose $s \in \Gamma(Y, \mathscr{L})$ is a nonvanishing global section and $\alpha(s)$ is not invertible. By Corollary 1.3, it must be that $\alpha(s)$ vanishes at some point x, say $\alpha(s)_x = m \in \mathfrak{m}_x$. Then $s = m\alpha_x^{-1}(1)$, so s vanishes, a contradiction.

Conversely, if s vanishes at x, say with $s_x = ms'$, then $\alpha(s)_x = \alpha_x(ms') = m\alpha_x(s')$, so $\alpha(s)$ vanishes and is not invertible. □

Corollary 3.4. *There is a canonical bijection between local trivializations on some open $U \subseteq Y$ of an invertible sheaf, and that sheaf's nonvanishing sections on U.*

The following shows that a converse to Corollary 2.6 holds in a useful class of cases.

Corollary 3.5. *Let $\phi : X \to Y$ be a morphism of locally ringed spaces, and $v \in \Gamma(Y, \mathscr{L})$. Then if $\phi^*(v)$ vanishes at some $x \in X$, we have that v vanishes at $\phi(x)$. In particular, if v is nonvanishing, then so is $\phi^*(v)$.*

Proof. Suppose that $\phi^*(v)$ vanishes at x. Then $v_{\phi(x)} \otimes 1 = m\sum_i (a_i \otimes b_i)$ for some $a_i \in \mathcal{L}_{\phi(x)}, b_i \in \mathcal{O}_{X,x}, m \in \mathfrak{m}_x$. Rearranging and using the fact that $\phi_x^\#$ is a local homomorphism, we find that for some $m' \in \mathfrak{m}_{\phi(x)}, a \in \mathcal{L}_{\phi(x)}$, we have $0 = (v_{\phi(x)} - m'a) \otimes 1$. Let $\psi : \mathscr{L}|_U \to \mathcal{O}_Y|_U$ be a local trivialization on some neighborhood of $\phi(x)$. Then $0 = (\psi_{\phi(x)}(v_{\phi(x)}) - m'\psi_{\phi(x)}(a)) \otimes 1$, which means that $\psi_{\phi(x)}(v_{\phi(x)}) - m'\psi_{\phi(x)}(a) \in \ker \phi^\#_{\phi(x)}$. But $\mathcal{O}_{Y,\phi(x)}$ is local, so $\ker \phi^\#_{\phi(x)} \subseteq \mathfrak{m}_{\phi(x)}$. We must therefore have that $\psi_{\phi(x)}(v_{\phi(x)}) \in \mathfrak{m}_{\phi(x)}$, say $\psi_{\phi(x)}(v_{\phi(x)}) = m'v'$, so applying $\psi^{-1}_{\phi(x)}$, we see that v vanishes at $\phi(x)$.

The second assertion is immediate from the first. □

Definition 3.6. Let $s_1, s_2 \in \Gamma(X, \mathscr{L})$, with s_2 nonzero. Let $\psi : \mathscr{L}|_{X_{s_2}} \to \mathcal{O}_X|_{X_{s_2}}$ be the local trivialization defined by s_2. We define $s_1/s_2 \in \Gamma(X_{s_2}, \mathcal{O}_X)$ to be $\psi(s_1)$.

Theorem 3.7. *Let $s_1, s_2 \in \Gamma(X, \mathscr{L})$. If $X_{s_2} \neq \emptyset$, then after restricting to X_{s_2} we have $(s_1/s_2)s_2 = s_1$. If $X_{s_1} \cap X_{s_2} \neq \emptyset$, after restricting to $X_{s_1} \cap X_{s_2}$ we have $(s_1/s_2) = (s_2/s_1)^{-1}$.*

Proof. To obtain the first equality, apply the trivialization taking s_2 to 1 to both sides. This is an isomorphism, the left hand side is now s_1/s_2 and by definition the right hand side is now s_1/s_2, so we must have had $(s_1/s_2)s_2 = s_1$. For the second equality, since $(s_1/s_2)s_2 = s_1$ and $(s_2/s_1)s_1 = s_2$, substituting, we obtain $(s_1/s_2)s_2 = (s_2/s_1)^{-1}s_2$, so applying the trivialization taking s_2 to 1 yields the result. □

4. Defining maps into projective space

We now have all the tools we need to prove our main theorem.

Theorem 4.1. *Let X be a scheme over a ring A.*

(1) *If $\phi : X \to \mathbb{P}^n_A$ is an A-morphism, then $\phi^*(\mathcal{O}(1))$ is an invertible sheaf on X, which is generated by the global sections $\phi^*(x_i), i = 0, \dots, n$.*
(2) *Let $\mathscr{L}$ be an invertible sheaf on X which is generated by nonzero global sections $s_0, \dots, s_n$. There exists a unique A-morphism $\phi : X \to \mathbb{P}^n_A$ for which there exists an isomorphism $\alpha : \mathscr{L} \to \phi^*(\mathcal{O}(1))$ with $\alpha(s_i) = \phi^*(x_i)$, and there is only one such α for this ϕ.*

Proof. Proposition 2.10 and Proposition 2.12 immediately yield the first part. We now prove the second part, beginning by constructing ϕ from our sections.

The open sets X_{s_i} cover X. By Lemma 3.2, each $\mathscr{L}_i := X_{s_i}$ is equipped with a trivialization $\psi_i : \mathscr{L}_i|_{X_{s_i}} \to \mathcal{O}_{X_{s_i}}$. Let U_i denote $D_+(x_i) \subset \mathbb{P}^n_A$. Let ϕ_i be the composition $X_{s_i} \xrightarrow{\zeta_i} \operatorname{Spec} A[x_0/x_i, \dots, x_n/x_i] \xhookrightarrow{\iota_i} \mathbb{P}^n_A$, with ζ_i induced by the homomorphism $A[x_0/x_i, \dots, x_n/x_i] \to \mathcal{O}_{X_{s_i}}$ sending x_j/x_i to s_j/s_i.

We want to show that $\phi_r|_{X_{s_r} \cap X_{s_t}} = \phi_t|_{X_{s_r} \cap X_{s_t}}$ for each r, t. The image of each of these restrictions is contained in $U_r \cap U_t$. Let

$$\tau_{rt} : \operatorname{Spec} A[x_0/x_r, \dots, x_n/x_r, (x_t/x_r)^{-1}] \to \operatorname{Spec} A[x_0/x_t, \dots, x_n/x_t, (x_r/x_t)^{-1}]$$

be the composition of the isomorphisms $\iota_r^{-1}(U_r \cap U_t) \to U_r \cap U_t \to \iota_t^{-1}(U_r \cap U_t)$. Then τ_{rt} is given by the identity. Thus we need to check that the ζ_i agree when extended to the same ring by localization, but this is the content of Theorem 3.7. This constructs ϕ, so we now need to check that α exists.

By Proposition 2.12, we have that $\mathcal{O}_X(1)$ is generated by the $\phi^*(x_i)$. Since $\mathcal{O}_X(1)$ is a pullback of an invertible sheaf, by Proposition 2.12 we have that $\mathcal{O}_X(1)$ is invertible. Since X_{s_i} maps into U_i, we have that $\phi^*(x_i)$ is nonvanishing, so defines a trivialization of $\mathcal{O}_X(1)|_{X_{s_i}}$. Letting

$\alpha_i : \mathscr{L}|_{X_{s_i}} \to \mathcal{O}_X(1)|_{X_{s_i}}$ be the composition of the maps taking s_i to 1 and 1 to $\phi^*(x_i)$, we see that α_i is an isomorphism from the fact that it is a composition of trivializations.

We now wish to show that the α_i are compatible, yielding the desired isomorphism $\alpha : \mathscr{L} \to \mathcal{O}_X(1)$. Pick any s_i, s_j. We assume all sections and morphisms are restricted to $X_{s_i} \cap X_{s_j}$. It is sufficient to show that $\alpha_i(s_i) = \alpha_j(s_i)$, or $\phi^*(x_i) = (s_i/s_j)\phi^*(x_j)$. On U_j we have an isomorphism $\mathcal{O}(1)|_{U_i} \to \mathcal{O}_{\mathbb{P}^n_A}|_{U_i}$ given by dividing by x_j. Applying the pullback of this isomorphism to both sides, the desired equation becomes $\phi^*(x_i/x_j) = s_i/s_j$. Since the left hand side is now the pullback of a section of the structure sheaf, this is $\phi^{\#}(x_i/x_j) = s_i/s_j$, which holds by definition. This constructs α. Further, it is easy to see from this construction that α was uniquely determined by the requirement $s_i \mapsto \phi^*(x_i)$.

Finally, suppose we have a map $\phi : X \to \mathbb{P}^n_A$ and an isomorphism $\alpha : \mathscr{L} \to \mathcal{O}_X(1)$, for which $\alpha(s_i) = \phi^*(x_i)$ for each i. We want to show that restricting ϕ to $\phi : X_{s_i} \to U_i$, we have $\phi^{\#}(x_j/x_i) = s_j/s_i$, or $\phi^{\#}(x_j) = (s_j/s_i)\phi^{\#}(x_i)$, which will specify the ring maps uniquely determining ϕ. Apply α to $s_j = (s_j/s_i)s_i$ to see that $\phi^*(x_j) = (s_j/s_i)\phi^*(x_i)$. The same trick of pulling back the division map shows us that $\phi^{\#}(x_j/x_i) = s_j/s_i$, as desired. $\square$