Module of Differentials Notes

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These notes draw from a variety of sources, including Hartshorne, Liu, a bit of Vakil and the Stacks project, and especially Eisenbud's *Commutative Algebra With a View Toward Algebraic Geometry*. The objective is to mishmash interesting elements from each of these into something light, explanatory, and extremely geometric, with opportunities for analogies with the theory of manifolds always being taken.

All rings are commutative and with identity.

1 Definition of the Module of Differentials

Definition 1.1. Let $\iota : R \to S$ be a ring map, and M be an S-module. An R-derivation $\varphi : S \to M$ is an R-linear map with $\operatorname{im}\iota \subseteq \ker\varphi$, and which satisfies the Leibniz rule $\varphi(gh) = g\varphi(h) + \varphi(g)h$. We let $\operatorname{Der}_R(S, M)$ be the S-module of R-derivations to M.

Example 1.2. In the above notation, let R by any ring, let $S = M = R[x, y]$, and let $\varphi : R[x, y] \to R[x, y]$ be the partial derivative map $f(x) \mapsto \partial f/\partial x$. This is an R-derivation. Notice that the elements of R serve the role of constant functions. It is also an $R[y]$-derivation.

Example 1.3. Let M be a manifold (by which I will always mean a smooth, real, finite dimensional manifold). Then M is equipped with a structure sheaf of $\mathbb{R}$-algebras $\mathcal{O}_M$ taking an open subset U to $C^\infty(U)$. A tangent vector at a point in M is a choice of direction and magnitude. However, given the many possible charts on M, it is not ***a priori*** clear that this is well defined. Worse, suppose we wish to define smooth vector fields on M: how can we encode the concept of smoothly choosing directions and magnitude at each point?

One way around this difficulty is to claim that the information of a vector v sticking out of a point p is the same as the information of how taking partial derivatives in the direction of v changes the value of functions at p. That is, fixing local coordinates on a chart U, we find that v is the same information as the map $\sigma_{p,v} : \mathcal{O}_M(U) \to \mathbb{R}$ which takes f to the partial derivative of f in the direction v, then evaluates at the point p. Taking different local coordinates, the derivative of the transition map between these coordinates yields another vector, which will turn out to yield exactly the same $\sigma_{p,v}$ if we repeat the procedure in our new coordinates. We can therefore define tangent vectors to be such $\sigma_{p,v}$. The result is a coordinate free encoding of the tangent vectors at p, which can be shown to agree with the explicit, coordinate based definitions.

This also allows us to encode a vector field X on U as an $\mathbb{R}$-derivation $X : \mathcal{O}_M(U) \to \mathcal{O}_M(U)$. To do this, we let $X(f)$ be the function so that if the vector associated with X at the point p is v, then $X(f)(p) = \sigma_{p,v}(f)$.

We can therefore associate another sheaf with M, the sheaf of derivations, which takes each U to the $\mathbb{R}$-module of derivations $X : \mathcal{O}_M(U) \to \mathcal{O}_M(U)$. This sheaf turns out to be locally free of rank $\dim M$, hence corresponds to a vector bundle, which is exactly the tangent bundle.

We now turn to some algebra. There is a universal object for R-derivations from S. We can describe it as follows.

Definition 1.4. Let $R \to S$ be a map of rings. Let $\mathbb{O}_{S/R}$, the *category of R-derivations*, be the category with objects (M, d_M), where M is an S-module and $\mathrm{d}_M : S \to M$ is an R-derivation (this category is not usually given a name or a symbol). The morphisms $(M, \mathrm{d}_M) \to (W, \mathrm{d}_W)$ of $\mathbb{O}_{S/R}$ are S-linear maps $f : M \to W$ so that the following triangle commutes

$$\begin{tikzcd} & S \arrow[dl, "\mathrm{d}_M"'] \arrow[dr, "\mathrm{d}_W"] & \\ M \arrow[rr, "f"] & & W \end{tikzcd} \tag{1}$$

Let $(\Omega_{S/R}, \mathrm{d})$, the module of relative differential forms of S over R, be the initial object of $\mathbb{O}_{S/R}$. We also call this the module of differentials or Kähler differentials of S over R. This is also called the Kähler module of S over R.

Less precisely, all R-derivations from S factor uniquely through $\Omega_{S/R}$. An equivalent formulation is that $\Omega_{S/R}$ represents the functor $\mathrm{Der}_R(S, -) : S\text{-}\mathbf{Mod} \to \mathbf{Set}$.

It may help to think of $\Omega_{S/R}$ as having the same relationship with derivations that the tensor product has with multilinear maps, or the wedge product has with alternating maps, etc. We show that $\Omega_{S/R}$ actually exists by constructing it: take a free module on the symbols $\{\mathrm{d}g : g \in S\}$ and then quotient down by the minimal relations necessary to induce the universal property. As a consequence $\mathrm{d} : S \to \Omega_{S/R}$ is surjective: everything in $\Omega_{S/R}$ is a linear combination over S of symbols of the form $\mathrm{d}g$ with $g \in S$.

Remark 1.5. Based on example 1.3, we should expect the tangent bundle to be encoded by some sheaf analogue of the module $\mathrm{Der}_R(S, S)$. We now wish to understand how to encode a cotangent bundle. In the manifold case, given a smooth function $f : M \to \mathbb{R}$, the derivative Df is a smooth map from the tangent bundle to $\mathbb{R}$ which is linear on the tangent space at each point. In other words, Df is a section of the cotangent bundle. Let T^* denote the sheaf of sections of the contangent bundle. Since $D(fg) = fDg + D(f)g$, we can regard D as a morphism of sheaves $D : \mathcal{O}_M \Rightarrow T^*$ which is given by an $\mathbb{R}$-derivation $D : \mathcal{O}_M(U) \to T^*(U)$ for each open set U. We thus obtain a unique map $\tilde{D} : \Omega_{\mathcal{O}_M(U)/\mathbb{R}} \to T^*(U)$ so that $D = \tilde{D} \circ \mathrm{d}$.

The cotangent bundle has sections which are locally of the form (for example) $fDg + qDh$, while $\Omega_{\mathcal{O}_M(U)/\mathbb{R}}$ has sections which are locally of the form (for example) $f\mathrm{d}g + q\mathrm{d}h$. We might conjecture that $\tilde{D}$ is an isomorphism. This conjecture is nearly correct, but the module of differentials only encodes relations between finite sums, so that, for example, letting $M = U = \mathbb{R}$, we do not have $\mathrm{d}e^t = e^t\mathrm{d}t$. Thus we can only get a 'purely algebraic' version of the cotangent bundle from examining the modules $\Omega_{\mathcal{O}_M(U)/\mathbb{R}}$. Since schemes are algebraic to start with this will not be an issue, so we will use modules of the form $\Omega_{S/R}$ to define the cotangent bundle.

2 Recipes

If this diagram of rings commutes

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R' \arrow[d] \\ S \arrow[r, "m"] & S' \end{tikzcd} \tag{2}$$

then by the universal property applied to the R-derivation $S \to S' \to \Omega_{S'/R'}$, we obtain the arrow making this diagram of S-modules commute:

$$\begin{tikzcd} S \arrow[r] \arrow[d, "\mathrm{d}"] & S' \arrow[d, "\mathrm{d}"] \\ \Omega_{S/R} \arrow[r, dashed, "\exists! \tau"] & \Omega_{S'/R'} \end{tikzcd} \tag{3}$$

Explicitly, the lower arrow is given by $\tau(f\mathrm{d}g) = m(f)\mathrm{d}(m(g))$. While this formula for τ doesn't mention R or R' at all, it requires the existence of a map $R \to R'$ to be well defined, and the internal structures of $\Omega_{S/R}$ and $\Omega_{S'/R'}$ depend on $R \to S$ and $R' \to S'$. The following result relates τ to the aforementioned maps.

Lemma 2.1. ***(Handy lemma)*** *In the above notation, if the map $S \to S'$ is surjective, then the kernel of the induced map τ is generated by those $\mathrm{d}a$ for which $m(a)$ lands in the image of R'.*

Proof. Suppose $\tau(\mathrm{d}a) = 0$. Then $\mathrm{d}m(a) = 0$ in $\Omega_{R'/S'}$, so in $\oplus_{w \in S'} S'\mathrm{d}w$ we can write $\mathrm{d}m(a)$ as a linear combination of the derivation relations. That is, there must be $r_i' \in R'$ and $s_i', t_i', x_i', y_i', c_{j,i}' \in S'$ so that we have a relation of formal symbols

$$\mathrm{d}m(a) = \sum_i c_{1,i}' \mathrm{d}r_i' + \sum_i c_{2,i}'(\mathrm{d}(s_i' + t_i') - \mathrm{d}(s_i') - \mathrm{d}(t_i')) + \sum_i c_{3,i}'(\mathrm{d}(x_i' y_i') - x_i' \mathrm{d}y_i' - y_i' \mathrm{d}x_i')$$

Since m is surjective, choosing preimages $s_i, t_i, x_i, y_i, c_{j,i} \in S$, we may write this as

$$\begin{aligned}\mathrm{d}m(a) = &\sum_i m(c_{1,i})\mathrm{d}m(r_i) + \sum_i m(c_{2,i})(\mathrm{d}(m(s_i + t_i)) - \mathrm{d}(m(s_i)) - \mathrm{d}(m(t_i))) + \\ &\sum_i m(c_{3,i})(\mathrm{d}(m(x_i y_i)) - m(x_i)\mathrm{d}m(y_i) - m(y_i)\mathrm{d}m(x_i))\end{aligned}$$

Now consider the map $\varphi : \oplus_{q \in S} S\mathrm{d}q \to \oplus_{w \in S'} S'\mathrm{d}w$ defined by $c\mathrm{d}q \mapsto m(c)\mathrm{d}m(q)$. We have

$$\varphi(\mathrm{d}a) = \varphi\left(\sum_i c_{1,i}\mathrm{d}r_i + \sum_i c_{2,i}(\mathrm{d}(s_i + t_i) - \mathrm{d}(s_i) - \mathrm{d}(t_i)) + \sum_i c_{3,i}(\mathrm{d}(x_i y_i) - x_i \mathrm{d}y_i - y_i \mathrm{d}x_i)\right)$$

It follows that for some $\sum_i \ell_i \mathrm{d}z_i \in \ker \varphi$, we have

$$\mathrm{d}a = \sum_i c_{1,i}\mathrm{d}r_i + \sum_i c_{2,i}(\mathrm{d}(s_i + t_i) - \mathrm{d}(s_i) - \mathrm{d}(t_i)) + \sum_i c_{3,i}(\mathrm{d}(x_i y_i) - x_i \mathrm{d}y_i - y_i \mathrm{d}x_i) + \sum_i \ell_i \mathrm{d}z_i$$

Taking the map $\oplus_{q \in S} S\mathrm{d}q \to \Omega_{S/R}$ which imposes derivation relations, in $\Omega_{S/R}$ we have that

$$\mathrm{d}a = \sum_i c_{1,i}\mathrm{d}r_i + \sum_i \ell_i \mathrm{d}z_i$$

Since each $m(r_i)$ is in the image of $R' \to S'$, it only remains to show that the sum $\sum_i \ell_i \mathrm{d}z_i$, when taken to $\Omega_{S/R}$, can be written as a linear combination of the desired differentials. For any $u \in S'$, let $\sum_{j \in J(u)} \ell_j \mathrm{d}z_j$ be the collection of terms in $\sum_i \ell_i \mathrm{d}z_i$ for which $m(z_i) = u$.

Since $\sum_i \ell_i \mathrm{d}z_i \in \ker \varphi$ and $\oplus_{w \in S'} S'w$ is free over S', we must have either $u = 0$ or $\sum_{j \in J(u)} m(\ell_j) = 0$. In the former case, we have that $\sum_{j \in J(u)} \ell_j \mathrm{d}z_j$ is in the desired form. In the latter case, reindexing the ℓ_j, we have $\ell_1 = -\ell_2 - \ell_3 - \ldots - \ell_n$. It follows that in $\Omega_{S/R}$ we have

$$\begin{aligned}\ell_1 \mathrm{d}z_1 + \cdots + \ell_n \mathrm{d}z_n &= -\ell_2 \mathrm{d}z_1 - \ldots - \ell_n \mathrm{d}z_1 + \sum_{i \neq 1} \ell_i \mathrm{d}z_i \\ &= \sum_{i \neq 1} \ell_i \mathrm{d}(z_i - z_1)\end{aligned}$$

which is in the desired form. □

2.1 Two exact sequences

If we have rings $A \to B \to C$, then

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & C & \end{tikzcd} \tag{4}$$

yields a map of C-modules $\eta : \Omega_{C/A} \to \Omega_{C/B}$. We likewise have a map of B-modules $\Omega_{B/A} \to \Omega_{C/A}$, which with the natural multiplication map from a tensor product defines a map of C-modules $C \otimes_B \Omega_{B/A} \to \Omega_{C/A}$. We can see that η is surjective. Applying the handy lemma shows that these maps assemble into an exact sequence of C-modules

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0 \tag{5}$$

A further recipe allows us to calculate the effect on the module of differentials of quotienting the larger of the defining rings. Any map of rings $A \to B$ and ideal $I \subseteq B$ yields an exact sequence

$$I/I^2 \xrightarrow{\delta} B/I \otimes_B \Omega_{B/A} \xrightarrow{\pi} \Omega_{(B/I)/A} \to 0 \tag{6}$$

with $\delta(b + I^2) = 1 \otimes \mathrm{d}b$. Applications of the Leibniz property show that δ is well defined and B/I-linear. The right three terms come from our previous exact sequence, with $C = B/I$. To see that $\operatorname{im}\delta = \ker \pi$, let $\overline{b}$ denote the quotient of an element $b \in B$. Then $\pi(\sum_i f_i \otimes h_i \mathrm{d}g_i) = \pi(\sum_i 1 \otimes f_i h_i \mathrm{d}g_i) = \sum_i \overline{f_i h_i} \mathrm{d}\overline{g_i}$. By the handy lemma this is zero exactly if we can write this so all the $\overline{f_i h_i}$ are zero (assume otherwise) or all the $\overline{g_i}$ are in the image of $A \to B \to B/I$. The dg_i which were not already zero must have $g_i \in I$ for this to be true, whence π vanishes exactly when $\sum_i 1 \otimes f_i h_i \mathrm{d}g_i \in \operatorname{im}\delta$.

Remark 2.2. We can illuminate the meaning of these sequences a little bit by again considering the case of manifolds. Given a submersion $\varphi : X \to Y$ of manifolds, we can define the relative tangent bundle $T_{X/Y}$ to be the subbundle given by the kernel of $D\varphi$ - in essence, by the regular value theorem, each level set of φ is a submanifold, and we take the union of the tangent vectors to these submanifolds. The relative cotangent bundle $T^*_{X/Y}$ is then the cotangent bundle of $T_{X/Y}$. In the case of a sequence of maps $X \xrightarrow{\varphi} Y \to Z$ which are all submersions, the following sequence of vector bundles over Z is exact

$$\varphi^*(T^*_{Y/Z}) \to T^*_{X/Z} \to T^*_{X/Y} \to 0 \tag{7}$$

where φ^* is the pullback functor on bundles.

On the other hand, if $X \hookrightarrow Y$ is a submanifold, we can define a bundle $N_{X/Y}$ of vectors normal to X inside of Y: it is the bundle fitting into the short exact sequence of vector bundles on X given by

$$0 \to T_X \to (T_Y)|_X \to N_{X/Y} \to 0 \tag{8}$$

That is, we restrict the tangent bundle of Y to X, then quotient by those vectors that are tangent to X, leaving only those which point in directions 'normal' to X. Dualizing this sequence yields a short exact sequence

$$0 \to N^*_{X/Y} \to (T^*_Y)|_X \to T^*_X \to 0 \tag{9}$$

Equation (7) and eq. (9) correspond to eq. (5) and eq. (6), respectively, except that when we work with less nice cases, we require correction factors like the tensor products and losing a zero in the second sequence.

Remark 2.2 motivates the naming of our sequences.

Definition 2.3. Equation (5) is the *cotangent sequence* and eq. (6) is the *conormal sequence.*

Example 2.4. In Remark 2.2, when $Y = X \times X$ and the map $X \xrightarrow{\Delta} X \times X$ is the diagonal, eq. (8) takes on particular significance. Here the sequence is

$$0 \to T_X \to (T_X \times T_X)|_{\operatorname{im}\Delta} \to N_{(X\times X)/X} \to 0 \tag{10}$$

Since the first map is $v \mapsto (v, v)$, we see that $N_{(X\times X)/X} \cong T_X$. Dualizing, $N^*_{(X\times X)/X} \cong T^*_X$.

2.2 The role of I/I^2

Remark 2.2 suggests that when we see an expression of the form I/I^2, we ought to draw analogies with the dual of the bundle of vectors normal to a submanifold. Example 2.4 suggests that for a ring S we ought to able to compute sections of our analogue of the cotangent bundle in terms of the diagonal of S.

To begin making this precise, let us fix some notation.

Definition 2.5. Let $R \to S$ be a map of rings. Let $m : S \otimes_R S \to S$ be multiplication. Let $I = \ker m$. Let $\mathrm{d} : S \to I/I^2$ be defined by $x \mapsto 1 \otimes x - x \otimes 1$. We call $(I/I^2, \mathrm{d})$, or for brevity just I/I^2, the *conormal module* of $R \to S$.

Throughout this section, let R, S, I and d be as in the above definition.

Here is a lemma that will be useful in our computations.

Lemma 2.6. *I consists of exactly those elements $\sum_i a_i \otimes b_i$ for which $\sum_i a_i b_i = 0$. Further, I is generated as an S-module by elements of the form $1 \otimes x - x \otimes 1$.*

Proof. The first statement is the definition. Now suppose we have $\sum_i a_i \otimes b_i \in I$. Then $\sum_i a_i \otimes b_i = \sum_i a_i \otimes b_i - \sum_i a_i b_i \otimes 1 = \sum a_i(1 \otimes b_i - b_i \otimes 1)$. □

There is a suggestive resemblance to an inner product in the above, but at least in these notes, nothing will come of it.

Theorem 2.7. *$(I/I^2, d)$ is the module of differentials of $R \to S$.*

Proof. We first note that I/I^2 has the structure of an S-module, induced by the S-module structure on $S \otimes_R S$. This is not multiplication into either factor, or as an S-module we would have $S \otimes_R S \cong S$ - it is multiplication into the first factor. Also note that by definition, $\sum_i a_i \otimes b_i \in I$ if and only if $\sum_i a_i b_i = 0$.

We first check that d is an R-derivation. For any $x, y \in S$, we have $\mathrm{d}(xy) = 1 \otimes xy - xy \otimes 1$ and $\mathrm{d}(x)y + x\mathrm{d}(y) = y(1 \otimes x - x \otimes 1) + x(1 \otimes y - y \otimes 1) = y \otimes x + x \otimes y - 2xy \otimes 1$. Thus $\mathrm{d}(xy) - \mathrm{d}(x)y - y\mathrm{d}(x) = 1 \otimes xy - y \otimes x - x \otimes y + xy \otimes 1$. We can write this as $(1 \otimes x - x \otimes 1)(1 \otimes y - y \otimes 1)$, so $\mathrm{d}(xy) = \mathrm{d}(x)y + y\mathrm{d}(x)$ in I/I^2.

Let (M, φ) be an object of $\mathbb{O}_{S/R}$. Let $\mathrm{d}_0 : S \to I$ be the map $x \mapsto 1 \otimes x - x \otimes 1$, and $\pi : I \to I/I^2$ be the quotient map. Then $\mathrm{d} = \pi \mathrm{d}_0$. Now let $\varphi^* : I \to M$ be defined by $x \otimes y \mapsto x\varphi(y)$. This is the composition of $\mathrm{id} \times \varphi : S \otimes_R S \to S \otimes_R M$ and multiplication, so is well defined. Notice that $\bar{\varphi}^* \mathrm{d}_0(x) = \bar{\varphi}^*(1 \otimes x) - x\bar{\varphi}^*(1 \otimes 1) = \varphi(x) - x\varphi(1) = \varphi(x)$.

Now suppose $\sum_i a_i \otimes b_i, \sum_j c_j \otimes e_j \in I$. Then

$$\begin{aligned}
\varphi^*\left(\left(\sum_i a_i \otimes b_i\right)\left(\sum_j c_j \otimes e_j\right)\right) &= \varphi^*\left(\sum_{ij} a_i c_j \otimes b_i e_j\right) \\
&= \sum_{ij} a_i c_j \varphi(b_i e_j) \\
&= \sum_{ij} (a_i c_j b_i \varphi(e_j) + a_i c_j e_j \varphi(b_i)) \\
&= \sum_j c_j \left(\sum_i a_i b_i\right) \varphi(e_j) + \sum_i a_i \left(\sum_j c_j e_j\right) \varphi(b_i) \\
&= \sum_j c_j \cdot 0 \cdot \varphi(e_j) + \sum_i a_i \cdot 0 \cdot \varphi(b_i) \\
&= 0
\end{aligned}$$

Since φ^* vanishes on I^2, we see that φ^* descends to an S-linear map $\bar{\varphi} : I/I^2 \to M$.

Suppose we have an S-linear map $\tau : I/I^2 \to M$ with $\varphi = \tau d$. From lemma 2.6, we know that τ is completely described by the values $\tau(1 \otimes x - x \otimes 1) = \tau d(x) = \varphi(x) = \bar{\varphi} d(x)$. It follows that $\tau = \bar{\varphi}$, demonstrating uniqueness. Thus $(I/I^2, \mathrm{d})$ is initial in $\mathbb{O}_{S/R}$, as desired. $\square$

The square of a kernel of a map from a tensor product is not necessarily all that easy to compute. For example, it is easier to check that $\Omega_{A[x_1,\ldots,x_n]/A}$ is the free $A[x_1, \ldots, x_n]$ module generated by differentials $\mathrm{d}x_i$ using the explicit construction, or the universal property, than by the above. The reason we bother with this construction is that it plays nicely with the sheaf structure on a scheme.

3 Extending the Construction to Schemes

Given all the work we've put in to defining the module of differentials of one ring over another, there can only really be one definition locally: given a morphism $\operatorname{Spec} S \to \operatorname{Spec} R$, we must have $\Omega_{\operatorname{Spec} S/\operatorname{Spec} R} = \widetilde{\Omega_{S/R}}$. Given a possibly non-affine scheme X, we want a sheaf that yields the above when restricted to open affines, and does so in a 'compatible' way. We could put the work into understanding what this means, or proceed with the following definition.

For us a locally closed embedding will be the composition of a closed embedding followed by an open embedding.

Lemma 3.1. *Let $\varphi : X \to Y$ be a morphism of schemes. Let $\Delta : X \to X \times_Y X$ be the diagonal morphism. Then Δ is a locally closed embedding.*

Proof. Let $V \subseteq Y$ and $W \subseteq X$ be open affine subsets, with $\varphi(W) \subseteq Y$. Then $W \times_Y W \subseteq X \times_Y X$ is open affine. We have that $\Delta^{-1}(W \times_Y W) = V$. We know the diagonal morphism is a closed embedding for affine schemes, so take the union of all such $W \times_Y W$. $\square$

Definition 3.2. Let $\varphi : X \to Y$ be a morphism of schemes. Let $\Delta : X \to X \times_Y X$ be the diagonal morphism. Let T be an open subset of $X \times_Y X$ into which Δ is a closed embedding. Let $\mathscr{I}$ be the sheaf of ideals for Δ in T. Then let $\Omega_{X/Y} := \Delta^*(\mathscr{I}/\mathscr{I}^2)$.

Notice that two choices T, T' of the open subset in the definition yield the same sheaf, since $\Delta^*(\mathscr{I}/\mathscr{I}^2)(U)$ is defined by taking a colimit over open sets containing $\Delta(U)$, which includes $T \cap T'$.

As a sanity check, we now show that when we reduce to the case of one affine scheme mapping into another, we get the right module back. If $U = \operatorname{Spec} B$ and $V \subseteq Y$ are open affine, with $\varphi(U) \subseteq V$, then $V \times_U V \hookrightarrow X \times_Y X$ is the open affine $\operatorname{Spec}(B \otimes_A B)$. By definition,

$\mathscr{I}(V \times_U V)$ is the kernel of $B \otimes_A B \to B$, so $\mathscr{I}/\mathscr{I}^2(V \times_U V)$ is the conormal module I/I^2 of $A \to B$, which is $\widetilde{\Omega_{B/A}}$. We know that the pullback of a module in the image of the $\widetilde{(-)}$ functor is given by the tensor product. That is, restricting Δ to $\Delta : V \to V \times_U V$, we have

$$\Delta^*(\mathscr{I}/\mathscr{I}^2)(V) = (\Omega_{B/A} \otimes_{B \otimes_A B} B)^{\sim}(V) = \Omega_{B/A} \otimes_{B \otimes_A B} B \cong \Omega_{B/A} \tag{11}$$

as desired.

This construction yields analogues of eq. (5) and eq. (6), which are as follows. Given a morphism of schemes $X \xrightarrow{f} Y \xrightarrow{g} Z$, the following sequence is exact

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0 \tag{12}$$

Given a morphisms of schemes $Z \hookrightarrow X \to Y$, where the first morphism is a closed embedding, we also obtain a canonical exact sequence of sheaves

$$\mathscr{I}_Z/\mathscr{I}_Z^2 \to \Omega_{X/Y} \otimes \mathcal{O}_Z \to \Omega_{Z/Y} \to 0 \tag{13}$$

where $\mathscr{I}_Z$ is the ideal sheaf of Z. The exactness of both these sequences reduces to the algebra we did earlier. We give these the same names we did earlier.

Definition 3.3. Equation (12) is the *cotangent sequence of sheaves* and eq. (13) is the *conormal sequence of sheaves.*

4 Smoothness

We will close with a brief discussion of smoothness, following Liu's *Algebraic Geometry and Arithmetic Curves.* Liu's definition of a variety is broad.

Definition 4.1. A *variety* $X \xrightarrow{\varphi} k$ is a scheme over a field k which is covered by finitely many open affine subsets U_i, each of which is of finite type over k via $\varphi|_{U_i}$.

We now define smoothness in two steps, beginning with the algebraically closed case, in which smoothness coincides with the familiar notion of regularity.

Definition 4.2. A point x in a scheme X is a *regular point* of X if $\mathcal{O}_{X,x}$ is a regular local ring.

Definition 4.3. Let X be a variety over an algebraically closed field. We define the *smooth* points of X to be the regular points.

Notice that this means we can check for smoothness at closed points using the Jacobian. The second step is to pass to general varieties.

Definition 4.4. If $X \to k$ is a variety, we say $x \in X$ is a ***smooth point*** if every $\bar{x} \in X_{\bar{k}}$ which lies above x is smooth according to definition 4.3.

The restate this, given the fiber product

$$\begin{tikzcd} X \times_k \operatorname{Spec} \bar{k} \arrow[r] \arrow[d, "\pi"] & \operatorname{Spec} \bar{k} \arrow[d] \\ X \arrow[r] & \operatorname{Spec} k \end{tikzcd} \tag{14}$$

we require that the points of $\pi^{-1}(x)$ each individually be smooth.

The idea here is that a scheme over a general field is a less geometric object than one over an algebraically closed field. For example, the "variety" in $\mathbb{A}^2_{\mathbb{R}}$ defined by $x^2 + y^2 = -1$ consists entirely of $\mathbb{C}$ points, which have no naive geometric interpretation. Upon passing to the algebraic closure, the vanishing locus of the same equation becomes a variety in the most classical sense, with explicit geometric features. This is reflected in the use of terms such as "geometrically irreducible."

We now define smoothness of general morphisms.

Definition 4.5. Let X and S be schemes, with S locally Noetherian, and let $X \xrightarrow{\varphi} S$ be a compact morphism of finite type. We say that φ is ***smooth at*** x if $X_{\varphi(x)}$ is smooth according to definition 4.4.

The reader should recall that $X_{\varphi(x)}$ is $\varphi^{-1}(\varphi(x))$ with a canonical scheme structure. Taking S to be a parameterization of a family of subschemes of X, this says that the family S parameterizes consists entirely of smooth schemes.

We'll wind up by mentioning with a couple more notable results, both of which you can find in Liu.

It isn't immediately clear what the connection is between smoothness and differential forms, but the following definition and theorem begin linking them together.

Definition 4.6. Let X be a scheme and $x \in X$. Then $\dim_x X$ is the least dimension among all neighborhoods of X.

Theorem 4.7. *Let X be a variety over a field k, and let $x \in X$. Then the following properties are equivalent:*

1. *X is smooth in a neighborhood of x.*

2. *$(\Omega_{X/k})_x$ is free of rank $\dim_x X$.*

Differential forms at a stalk can be thought of as infinitesimal shadows of coordinate functions, so one ought to interpret this as suggesting there are the correct number of independent coordinate functions near x.

The second result, which continues developing this intuition, is revealing of the nature of smoothness over non-field bases. It uses the notion of étale morphisms, which we won't define but which are morally local isomorphisms (here 'local' means with respect to the domain and on a very, very small neighborhood, potentially too small for the Zariski topology to describe. Think of a covering space).

Theorem 4.8. *Let $X \xrightarrow{f} S$ be a morphism to a locally Noetherian scheme, smooth at a point $x \in X$. Then there exists a number n and a neighborhood U of x so that we have the following factorization*

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[dr, "f|_U"'] & \mathbb{A}^n_S \arrow[d] \\ & S \end{tikzcd} \tag{15}$$

Further, φ is étale at x.

We should interpret this as follows: on a tiny neighborhood about x, potentially smaller than U, we can think of f as given by projection of a trivial bundle down to S.

We will conclude by describing how φ comes about, connecting this result to the prior theorem. By theorem 4.7, choose a basis $\{\mathrm{d}f_1, \ldots, \mathrm{d}f_n\}$ for $(\Omega_{X/k})_x$ over $\mathcal{O}_{X,x}$. Choose U about x to be open affine, say $\operatorname{Spec} R$, and small enough that the $\mathrm{d}f_i$ all lift to global sections of $\mathcal{O}_U$. Then R is an A-algebra, and $x_i \mapsto f_i$ defines a map of rings $A[x_1, \ldots, x_n] \to R$, thus a morphism $\varphi : U \to \mathbb{A}^n_A$ through which f factors.