# A PROOF OF CHEVALLEY'S THEOREM, FLAT MORPHISMS ARE OPEN

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This is essentially an exposition of a few exercises in Hartshorne.

There are two main statements we will prove. Recall that a *constructible subset* of a topological space is a finite union of locally closed sets. That is, $S$ is constructible if $S$ is of the form $(U_1 \cap K_1) \cup (U_2 \cap K_2) \cup \cdots \cup (U_n \cap K_n)$, where all the $U_i$ are open and the $K_i$ are closed.

**Theorem 0.1.** *Let $\varphi : X \to Y$ be a morphism of finite type between Noetherian schemes. Then the image of any constructible subset is constructible.*

It can be shown that equivalently, constructible sets are exactly the boolean algebra generated by open sets, finite unions, and complements. This gives them significance in applications of model theory to algebraic geometry (in affine space over an algebraically closed field, they are exactly the quantifier free definable sets). They also permit a fancy generalization in constructible sheaves, which are significant in the theory of étale cohomology.

The above statement is sufficient for maps between varieties, but the theorem can be proven more generally than will be seen here. The broader statement is as follows:

**Theorem 0.2.** *Let $\varphi : X \to Y$ be a finitely presented morphism of schemes. Then $\varphi$ takes locally constructible*[1] *subsets to locally constructible subsets.*

Having proved Chevallay's theorem, we will provide a straightforward application which may illuminate flatness a little bit.

**Theorem 0.3.** *Let $\varphi : X \to Y$ be a flat morphism of finite type between Noetherian schemes. Then $\varphi$ is a open map.*

## 1. Noetherian Induction

Our proof will be by Noetherian induction, which we will briefly review.

**Proposition 1.1.** *Let $P$ be a poset satisfying the ascending chain condition. Let $\mathscr{P}$ be a property of elements of $P$ satisfied by at least one element. Then there is some element maximal among those satisfying $\mathscr{P}$.*

[1]This requires slightly changing the definition of constructability, in a way we won't discuss. The term "locally constructible" means "constructible on each set of an open cover"

*Proof.* Let $P'$ be the subposet containing all elements satisfying $\mathscr{P}$. Since $P$ satisfies the ascending chain condition, so does $P'$, so by Zorn's lemma the desired element exists. □

**Proposition 1.2.** *Let $P$ be a poset satisfying the ascending chain condition, and let $\mathscr{P}$ be a property of elements of $P$ such that for every $x \in P$, if all $y \in P$ with $y > x$ satisfy $\mathscr{P}$, then $x$ satisfies $\mathscr{P}$. Then every element of $P$ satisfies $\mathscr{P}$.*

*Proof.* Suppose otherwise. By the previous proposition there exists some element maximal among those not satisfying $\mathscr{P}$, yielding a contradiction. □

**Corollary 1.3.** *Let $X$ be a Noetherian topological space and $\mathscr{P}$ be a property of closed sets such that for every closed subset $K$, if all closed $K' \subset K$ satisfy $\mathscr{P}$, then $K$ satisfies $\mathscr{P}$. Then $\mathscr{P}$ holds for every closed subset of $X$, and in particular holds for $X$.*

## 2. Generic Freeness

Remember that we ought to think of an $\mathcal{O}_X$-module as being something like the sheaf of sections of a vector bundle, except that the fibers may be attached together in strange ways or suddenly change their dimension. Quasicoherent sheaves are those which arise algebraically, while coherent sheaves are, in addition, conceptually finite dimensional. It is a significant fact that for coherent sheaves on integral schemes, not only will the 'weird fibers' only occur along a relatively small set, so that we can bundle-ify the sheaf by cutting out a hypersurface, but that the relations which prevent the resulting bundle from being trivial can also be removed by cutting out a hypersurface (for a real manifold example of this last fact, consider how removing a single point from $S^1$ allows us to untwist the mobius band over it, yielding a rectangle). This phenomenon is called *generic freeness*; we present a special case of it.

Recall that $\kappa(x)$ denotes the residue field of a point $x$.

**Definition 2.1.** Let $X$ be a scheme and $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $x \in X$. Then the *fiber of $\mathcal{F}$ over $x$*, which we denote by $\kappa_{\mathcal{F}}(x)$, is the $\kappa(x)$ vector space $\mathcal{F}_x \otimes_{\mathcal{O}_{x,x}} \kappa(x)$.

Notice that if $\mathcal{F}$ were a geometrically realized vector bundle or something similar over $X$, algebraically $\kappa_{\mathcal{F}}(x)$ is what we would get by taking the fiber over $x$.

**Theorem 2.2.** *Let $\mathcal{M}$ be a coherent sheaf on a locally Noetherian scheme $x$. Let $x \in X$ and let $e_1, \ldots, e_n$ be generators of $\kappa_{\mathcal{F}}(x)$. Then we can lift the $e_i$ to an open neighborhood $U$ of $x$ such that the $e_i$ generate the stalks of every point in $U$.*

*Proof.* By Nakayama's lemma, the $e_i$ may be lifted to a generating set for $\mathcal{F}_x$, which we then lift to a set of sections also denoted by $e_i$ on an open affine neighborhood $\operatorname{Spec} A$ of $x$. Now on

$\operatorname{Spec} A$ we have that $\mathcal{F} \cong \widetilde{M}$ for some finitely generated $A$-module $M$, so we can pick generators $q_1, \ldots, q_m$ of $\mathcal{F}(Spec(A))$. Note that localizing brings generating sets of modules to generating sets (we can see this by taking a presentation, then using right exactness of the tensor product and the fact that localizing free modules gives free modules). Thus it is sufficient to write the $q_i$ as linear combinations of the $e_i$ on some neighborhood $U$ of $x$: we will have that the $q_i$ are in $(e_1, \ldots, e_n)$, so $(e_1, \ldots, e_n) = \mathcal{F}(U)$, so the $e_i$ generate the stalks of all points in $U$.

In $\mathcal{F}_x$, we can write $q_i = \sum_j c_{ij} e_j$ for some $c_{ij}$. Shrink our open set until we have lifts of all these $c_{ij}$, then shrink again until $q_i - \sum_j c_{ij} e_j = 0$ for all $i$. This is our $U$. □

**Corollary 2.3.** *(Upper semicontinuity of dimension) For all $y \in U$, we have* $\dim_{\kappa(y)} \mathcal{F}_y \otimes_{\mathcal{O}_{X,y}} \kappa(y) \leq \dim_{\kappa(x)} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$.

To see an example of the corollary in action, consider the tangent bundle of a curve with a self intersection: if the curve is smooth, it's also smooth at nearby points, and the fibers of the bundle are of dimension one. But at the point of self intersection, the dimension of the tangent bundle jumps.

Now here is our form of generic freeness.

**Theorem 2.4.** *Let $A$ be an integral domain and let $M$ be a finitely generated $A$-module. There exists some nonzero $f \in A$ such that $M_f$ is free of finite rank.*

*Proof.* Let $\mathcal{F} = \widetilde{M}$. Let $\eta \in \operatorname{Spec} A$ be the generic point. By the previous theorem we may lift a basis for $e_1, \ldots, e_n$ of $\kappa_{\mathcal{F}}(\eta)$ to a generating set on some neighborhood $U$ of $\eta$. Now suppose there exists a relation $c_1 e_1 + \cdots + c_n e_n$. Passing to Frac$A$ would kill all the $c_i$, since the $e_i$ are linearly independent in $\mathcal{O}_{x,\eta}$, but since $A$ is an integral domain this shows the $c_i$ are all zero. Passing to some basic open $D(f)$ shows that $M_f$ is free. □

## 3. Chevalley's Theorem

We begin with some algebra.

**Lemma 3.1.** *Let $B$ be an integral domain and $A$ a subring $A \hookrightarrow B$ such that $B$ is finite over $A$. Let $b \in B$ be nonzero. There exists some $a \in A$ with the following property. If $\varphi^* : A \to \overline{k}$ is a homomorphism to an algebraically closed field with $\varphi(a) \neq 0$, then $\varphi^*$ extends to $B$ in such a way that $\varphi^*(b) = 0$. Equivalently, every morphism $\varphi : \operatorname{Spec} \overline{k} \to \operatorname{Spec} A \backslash V(a)$ lifts to some $\varphi : \operatorname{Spec} \overline{k} \to \operatorname{Spec} B \backslash V(b)$.*

*Proof.* By generic freeness, we may choose some nonzero $f \in A$ such that $B[f^{-1}]$ is free over $A[f^{-1}]$. Choosing a basis, multiplying by $b$ is an $A[f^{-1}]$-linear transformation of $B[f^{-1}]$, encoded by some matrix with a determinant $r/f^n \in A[f^{-1}]$, where $r \in A$. Since $b \neq 0$ and $B[f^{-1}]$ is an integral domain, we must have that $r \neq 0$. Now let $a = rf$.

Supposing we have some $\varphi^*$ as in the statement, let $x$ be the point picked out by $\varphi$. Since $x \notin V(r)$, it cannot be that $b$ restricts to zero on the fiber over $x$, whence there is some point $x'$ in the fiber which is not in $V(b)$. By Zariski's lemma $\kappa(x')$ is a finite extension of $\kappa(x)$, so $\varphi^*$ extends to an embedding of $\kappa(x')$ into $\overline{k}$, so $\varphi$ lifts to $x'$, as desired. □

**Proposition 3.2.** *The same result holds in the situation of the above lemma when $B$ is of finite type over $A$.*

*Proof.* We proceed by induction on the number of generators $n$ of $B$. If $n = 0$ there is nothing to prove.

If $n = 1$, say $B = A[t]$, there are two cases. If $t$ satisfies no relations over $A$, then writing $b = \sum_i c_i t^i$, we can extend any $\varphi^*$ by sending $t$ to any non-root of $\sum_i \varphi^*(c_i)x^i$. If $t$ does satisfy a relation, say $c_0 t^m + c_1 t^{m-1} + \cdots + c_m = 0$, then $B[c_0^{-1}]$ is a finite module over $A[c_0^{-1}]$, so we can apply the lemma to produce an $a'$ so that we can extend any $\varphi^*$ which does not send $c_0$ or $a'$ to zero. Thus the desired $a$ is $c_0 a'$.

Now suppose the result holds for all $A[t_1, \ldots, t_{n-1}]$. Since $B = A[t_1, \ldots, t_n]$ is of finite type over $A[t_1, \ldots, t_{n-1}]$, we can apply the $n = 1$ case to find some $a' \in A[t_1, \ldots, t_{n-1}]$, then apply the induction hypothesis with $a'$ to find an $a \in A$. Now if $\varphi^*(a) \neq 0$, then we can extend $\varphi^*$ to $A[t_1, \ldots, t_{n-1}]$ so that $\varphi^*(a') \neq 0$, and we can therefore extend to $B$ so that $\varphi^*(b) \neq 0$. □

**Corollary 3.3.** *Let $\varphi : X \to Y$ be a dominant morphism of finite type of integral schemes. Then* $\operatorname{im} \varphi$ *contains a nonempty open set.*

*Proof.* Let $\operatorname{Spec} A$ be an affine open subset of $Y$ with preimage $\operatorname{Spec} B$, where $B$ is finitely generated over $A$. Apply the previous result with any $b$ to find that all of $D(a)$ is in the image of $\varphi$. □

We can now prove Chevalley's theorem.

*Proof.* We may assume that $X$ and $Y$ are reduced, since the question is topological and nilpotents do not affect the topology. Let $S \subseteq X$ be constructible, say $S = (U_1 \cap K_1) \cup (U_2 \cap K_2) \cup \cdots \cup (U_n \cap K_n)$, where the $U_i$ are open and, without loss of generality, the $K_i$ are irreducible. Then $U_i \cap K_i$ is an open subscheme of $K_i$ with the reduced induced structure, and if the image of each $U_i \cap K_i$ is constructible so is the image of $S$, so we may as well assume $X = U_i \cap K_i$. Note that in this case $X$ is integral.

Now let $\mathscr{P}$ be the property on closed sets that all morphisms from such an $X$ to the closed set have a constructible image. We will apply Noetherian induction. Pick a closed set $K$ so that all proper closed subsets of $K$ have property $\mathscr{P}$. If $K$ is empty, there is nothing to show.

Suppose otherwise, and that $X$ maps into $K$. If $K$ is reducible, write $K = K_1 \cup \cdots \cup K_n$. Then $\operatorname{im} \varphi = \operatorname{im} \varphi|_{\varphi^{-1}(K_1)} \cup \cdots \cup \operatorname{im} \varphi|_{\varphi^{-1}(K_n)}$, so $\mathscr{P}$ holds.

Now suppose that $K$ is irreducible. If the image of $\varphi$ in $K$ is not dense, then taking the closure $T$ of the image, since $T$ is a proper subset of $K$, by hypothesis $\varphi$ has a constructible image. If the image of $\varphi$ is dense, then applying the previous corollary, we find $\operatorname{im} \varphi$ contains an open set $U$. Now $\operatorname{im} \varphi = U \cup (\operatorname{im} \varphi \cap (K \backslash U))$. By hypothesis $\operatorname{im} \varphi \cap (K \backslash U)$ is constructible, and $U$ is constructible by virtue of being open, so $\operatorname{im} \varphi$ is constructible. □

This result immediately lets us strengthen the corollary we used to prove it.

**Proposition 3.4.** *Let $\varphi : X \to Y$ be a dominant morphism of finite type between Noetherian schemes. Then $\operatorname{im} \varphi$ contains a dense open set.*

*Proof.* It is enough to prove this when $Y$ is irreducible. Let $\operatorname{im} \varphi = \cup_{i=1}^{n} (U_i \cap K_i)$, with the $U_i$ open, the $K_i$ closed, and none of the intersections empty. If none of the $K_i = Y$ then since $\operatorname{im} \varphi \subseteq \cup_{i=1}^{n} K_i$ we contradict the dominance of $\varphi$. □

## 4. Openness of Flat Maps

We will provide an application by proving that flat morphisms of finite type between Noetherian schemes are open (i.e. they take open sets to open sets).

**Lemma 4.1.** *Let $\varphi : X \to Y$ be a flat morphism of schemes, with $Y$ integral. Then $\varphi$ is dominant.*

*Proof.* Without loss of generality, we may assume that $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$. Pick any nonzero $f \in A$. Then $0 \to A \xrightarrow{f} A$ pulls back to $0 \to B \xrightarrow{\varphi^*(f)} B$, so $\varphi^*$ must be injective, whence $\varphi$ is dominant. □

**Definition 4.2.** Let $X$ be a Noetherian scheme. If $p, F$ are points in $X$ and $p \in \overline{\{F\}}$, then we say that $p$ is a *specialization* of $F$ and that $F$ is a *generization* of $p$.

The reason for this naming is that we should think of restricting to $\overline{\{F\}}$ as imposing some relation on the inputs of functions, and we should think of restricting to $\overline{\{p\}}$ as imposing a more restrictive relation. We can also think of this as partial function evaluation. For example, in affine two space, passing to $k[x, y]/(x - y^2)$ represents partially evaluating functions, while passing to $k[x, y]/(x - 1, y - 1)$ represents fully evaluating functions.

**Lemma 4.3.** *Let $T$ be a subset of a Noetherian scheme $X$. Then $T$ is closed if and only if it is constructible and closed under specialization, and open if and only if it is constructible and closed under generization.*

*Proof.* If $T$ is open or closed, certainly $T$ is constructible. Pick any $p \in U$. If $T$ is open, then if $p \in \overline{\{F\}}$, and $U$ does not include $F$, then $(X \backslash U) \cap \overline{\{F\}}$ is a strictly smaller closed set containing $F$, which is a contradiction. If $T$ is closed, then $T$ contains $\overline{\{p\}}$, hence any specialization of $p$.

Conversely, let $T = \cup_{i=1}^{n}(U_i \cap K_i)$, where the $U_i$ are open and the $K_i$ are closed. If $T$ is closed under specialization, then by the same reasoning as above, each nonempty $U_i \cap K_i$ contains all the generic points of $K_i$, hence by closure under specialization contains $K_i$, so that $T = \cup_{i=1}^{n} K_i$ is closed. If $T$ is closed under generization, then $X \backslash T$ is constructible and necessarily closed under specialization, so as just shown, is closed. Thus $T$ is open. □

**Theorem 4.4.** *Let $\varphi : X \to Y$ be a flat morphism of finite type between Noetherian schemes. Then $\varphi$ is an open map.*

*Proof.* It is enough to show that $\operatorname{im}\varphi$ is open: the same reasoning will work for any open subscheme $U$ of $X$, since the composition $U \hookrightarrow X \xrightarrow{\varphi} Y$ is flat. We know from Chevalley's theorem $\operatorname{im}\varphi$ is constructible. By the previous lemma it is sufficient to prove that $\operatorname{im}\varphi$ is closed under generization. Let $x \in \operatorname{im}\varphi$ be in $K = \overline{\{x'\}}$, where we give $K$ its reduced induced structure. Flatness is preserved under base change and so $\varphi_K : X_K \to K$ is flat. Since $K$ is integral it follows from Lemma 4.1 that $\varphi_K$ is dominant. By Proposition 3.4 this means that the image contains an open set, and therefore contains $x'$. □